IJTASE
ISSN: 2146 - 9466
www.ijtase.net
International Journal of New Trends in Arts, Sports &Science Education - 2021, volume 10, issue 1

THE CONTINUITY AND THE IMPORTANT THEOREMS IN THE TOPOLOGICAL KNOWLEDGE SPACES

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Received: 18-11-2020 **Accepted:** 30-12-2020 **Published:** 31-01-2021

Abstract
In this study my goal is to define the concept of continuity and to uncover its important in the topological knowledge space. Moreover, I introduce pedagogically sound concept of continuity for the topological knowledge structure modeling a students' learning and the important theorems are proofed. Indeed, some definitions are presented and theorems are proofed. Consequently, it is proofed that in the topological knowledge spaces as a result of continuity well-graded, learning smooth and accessible have been kept in continuity. Thus, a minimal set of requirements for a knowledge structure whose emphasis is the modeling of students' learning can be described as a minimal set of requirements for making up a different knowledge space.
Keywords: Continuity, topological knowledge, topological spaces

Introduction

At the inception, the ambition of Knowledge Space Theory was to offer a sound combinatoric framework to design an efficient machine for the assessment of knowledge (Doignon & Famagne, 1985; Doignon & Famagne, 1999). The field of knowledge, or "domain", is simply identified with the full collection of items. A student's mastery, or "knowledge state", is the subset of items that she can answer correctly. The family of feasible knowledge states defines the "knowledge structure" (Cosyn & Uzun, 2009). Topological knowledge space is to enable the evaluation of knowledge state topologically. The continuity in topological knowledge space is to enable to translate the different knowledge spaces. Thus, a minimal set of requirements for a knowledge structure whose emphasis is the modeling of students' learning can be described as a minimal set of requirements for making up a different knowledge space (Cosyn & Uzun, 2009). For this definition, it is required that there exists a continuity function have kept features well-graded, learning smooth and accessible in topological knowledge space.

Definition 1.1. A knowledge structure is a pair (X,τ) in which $X \neq \phi$ and τ is a family of subsets of X, containing at least X and the empty set ϕ. In this situation τ is called the indiscrete knowledge structure.

1) A knowledge structure τ is called under union

(that is, for any two states $\tau_1, \tau_2 \in \tau$, we have $\tau_1 \cup \tau_2 \in \tau$).

2) For any finite subsets $\tau_i \quad (i = 1,2,..., n)$, we have $\cap \tau_i \in \tau$

If there are the above two conditions on knowledge structures, we say that τ is a topology for X and a pair (X,τ) is called topological knowledge space.

The two conditions, which are of a different nature, have played a role in the assessment procedures and the probabilistic learning models developed by Falmagne (Falmagne, 1989; Falmagne 1993).

Definition 1.2. A knowledge space (X,2) is well-graded if, for any $A, B \in \tau$, there exist a finite sequence of sets $A = \tau_0, \tau_1,..., \tau_n = B$ in τ such that

$i) |A \Delta B| = k,$ and

$ii) |\tau_{i-1} \Delta \tau_i| = 1, \quad i = 1,2,..., k$

Where Δ denotes the standard symmetric difference between sets. The sequence of sets $A = \tau_0, \tau_1, ..., \tau_n = B$ satisfying these conditions is called a tight path between A and B (Cosyn & Uzun, 2009). The condition of closure under union has been main requirement on a knowledge structure. It is a fairly reasonable expectation on the family of knowledge states: If there is a student knowing exactly the items in K and another knowing exactly the items in K and another knowing exactly the items in L then we may expect that there could be a student knowing exactly the items in $K \cup L$, and the two statements could talk to other students one by one and he brings his knowledge state to $K \cap L$. The knowledge structure is transformed into topology by the method allows them. Well-graded has been the other requirement on the topological knowledge space.

The Continuity

There is a considerable number of auxiliary concepts which play a role in connection with topological knowledge spaces**.** One of the most important concepts is the continuity which we shall need is included in this section on the topological knowledge space. Some learning situation can be explained on the topological knowledge spaces by the continuous surmise mapping. First of all, we shall define some definition.

Definition 2.1. Let (X, τ) be a topological knowledge space. For any knowledge state A

$\tau_k \subset A$ $(k = 1,2,3,...,n)$ the greatest subset τ_k is called inner of A and is shown by $\overset{o}{A}$.

$A \subset \tau_k$ $(k = 1,2,3,...,n)$ the smallest closed set τ_k is called closure of A and is shown by $\bar{A}$.

Definition 2.2. Let (X, τ) be a topological knowledge space, for any $\tau_k \in \tau$, the element τ_k is called open set on this space. The complement of set A is called closed set.

Definition 2.1. implies that the inner of A represents the most advanced material that a student in state A masters, and the closure of A represents the material that the same student is ready to learn.

A knowledge structure is learning smooth if, for any two states $K \subseteq L$, there exists a tight path from K to L (Axiom 2 in Cosyn & Uzun, 2009).

Definition 2.3. A knowledge structure is accessible if, for any state K, there exists a tight path from ϕ to K.

Here after, it is shown that the continuity allows well-graded, learning smooth, and accessible on topological knowledge space.

Definition 2.4. Let (X_1, τ_1), (X_2, τ_2) be topological knowledge spaces and let $f : X_1 \to X_2$ give surmise mapping.

A surmise mapping f of a topological knowledge space (X_1, τ_1) into a topological knowledge space (X_2, τ_2) is continuous if and only if the inverse image of any open subset of X_2 is an open subset of X_1.

A more interesting property of knowledge structure is that of well-gradeness (Falmagne & Doignon, 1988 and Falmagne, 1989)

Definition 2.5. A knowledge structure K on X is said to be well-graded if for any $k \subseteq K$,

$k \neq \emptyset \Rightarrow k \setminus \{x\} \in K, x \in k,$

$k \neq X \Rightarrow k \cup \{y\} \in K, y \in X \setminus k,$ (Koppen, 1998).

Theorems

Some theorems are to explain the role of continuity. Those shall help to explain well-graded, smooth learning and accessible in two different topological knowledge spaces.

Theorem 3.1. (X_1,τ_1), (X_2,τ_2) are topological knowledge spaces and let τ surmise mapping f be continuous and bijective $f:(X_1,\tau_1)\to(X_2,\tau_2)$, τ_1 is well-graded if τ_2 is well-graded.

Proof. $\varnothing\in\tau_2\Rightarrow\varnothing\in\tau_1$

i) $K_0\in\tau_2\ and\quad K_0\cup\{q\}=K_1\in\tau_2,\ q\in X_2\setminus K_0$

$\Rightarrow L_0=f^{-1}(K_0)\in\tau_1$ and

$f^{-1}(K_0\cup\{q\})=f^{-1}(K_0)\cup f^{-1}(\{q\})=L_0\cup\{t\}\in\tau_1$

$\Rightarrow L_0\in\tau_1\ and\quad L_0\cup\{t\}\in\tau_1,\qquad t\in X_1\setminus \mathrm{L}_0\Rightarrow\tau_1$

ii) $K_0\setminus\{\mathrm{x}\}\in\tau_2\ ,\ x\in K_0$

$\Rightarrow L_0=f^{-1}(K_0)\in\tau_1$ and $f^{-1}(x)=y\in L_0$

$$\Rightarrow f^{-1}(K_0\setminus\{\mathrm{x}\})=f^{-1}\left(K_0\cap\{x\}'\right)$$
$$=f^{-1}(K_0)\cap f^{-1}\left(\{x\}'\right)$$
$$=L_0\setminus \mathrm{y}\in\tau_1.$$

Hence, τ_1 is well-graded.

Let the knowledge states in the two different terms have got a bijective structure. Under this initial condition, the theorem means that we think that, if students' topological knowledge space is well-graded at the second term, their topological knowledge space is well-graded at the first term. It means that. For instance, when we investigate the concept of length and area. If students' knowledge state is well-graded in the areas of square, rectangle, and then their knowledge state is well-graded in the length of segment.

Theorem 3.2. (X_1,τ_1), (X_2,τ_2) are topological knowledge spaces and let a surmise mapping f be continuous and bijective, τ_1 is learning smooth, if τ_2 is learning smooth.

Proof. $K,L\in\tau_2$, for any two states $K\subseteq L$, there exists a tight path from K to L.

$K=A_0,A_1,\ldots,A_n=L$

$|K\Delta L|=n$

$|A_{i-1}\Delta A_i|=1,\ i=1,2,\ldots,n$ and $A_0,A_1,\ldots,A_n\in\tau_2$

By the continuous maps f, there exists

$f^{-1}(A_i)=B_i\in\tau_1,\ i=0,1,2,\ldots,n.$

Since the surmise mapping f is bijective

$$f^{-1}(A_{i-1}\Delta A_i)=f^{-1}[(A_{i-1}\setminus A_i)\cup(A_i\setminus A_{i-1})]$$
$$=f^{-1}(A_{i-1}\setminus A_i)\cup f^{-1}(A_i\setminus A_{i-1})$$
$$=f^{-1}\left(A_{i-1}\cap A_i'\right)\cup f^{-1}\left(A_i\cap A_{i-1}'\right)$$

$$=\left[f^{-1}(A_{i-1})\cap f^{-1}\left(A_i'\right)\right]\cup\left[f^{-1}(A_i)\cap f^{-1}\left(A_{i-1}'\right)\right]$$

$$=\left(B_{i-1}\cap B_i'\right)\cup\left(B_i\cap B_{i-1}'\right)$$

$$=(B_{i-1}\setminus B_i)\cup(B_i\setminus B_{i-1})$$

$$=B_{i-1}\Delta B_i$$

$$|A_{i-1}\Delta A_i|=1 \quad\Rightarrow\quad |B_{i-1}\Delta B_i|=1$$

$$|B_0\Delta B_n|=n \;\; K'=B_0,B_1,...,B_n=L'$$

$$K'=f^{-1}(K)\subseteq f^{-1}(L)=L'$$

For any two states $K'\subseteq L'$ there exists $K',L'\in\tau_1$. It shows that there exists a tight path from K' to L'. On the other hand, τ_1 is a learning smooth? The learning smooth ensures that student can learn new items on time whatever her mastery is. She always might be in stage of learning. Students can smooth in their learning with the modeling, which is a technique of modeling, if their mastery is a learning smooth. They can get better the level of developing knowledge state whichever her mastery is.

Theorem 3.3. Let (X_1,τ_1) and (X_2,τ_2) be topological knowledge spaces. We think that a surmise mapping $f:(X_1,\tau_1)\to(X_2,\tau_2)$ is continuous and bijective. If τ_2 is accessible, τ_1 is accessible.

Proof. For any $K\in\tau_2$ and $\phi\in\tau_2$. There exists a sequence of knowledge states $\emptyset, K_0, K_1,..., K_n$

$f^{-1}(\phi)=\phi$ and $f^{-1}(K)=L$

$f^{-1}(K_{i-1}\Delta K_i)=L_{i-1}\Delta L_i$, As a surmise mapping f is a continuous and bijective, it can be obtained that

$|K_{i-1}\Delta K_i|=1=|L_{i-1}\Delta L_i|$.

This shows us that for any $L\in\tau_1$, there exists a tight path from ϕ to L. Then τ_1 is accessible. Accesssibleness expresses us that a student can progress in knowledge space by utilizing from different processing while she obtains knowledge states. After these theorems, we can reach following results.

Results 3.1. The topological knowledge spaces $(X_1,\tau_1)$and (X_2,τ_2), well-gradedness covers learning smooth and accessible.

Results 3.2. If the topological knowledge spaces $(X_1,\tau_1)$and (X_2,τ_2) are learning smooth, one is well-graded and accessible but the other one may not be well-graded and accessible.

Homeomorphic Space and Homeomorphism

We need to know the definition of homeomorphic space and homeomorphism. Thus, we can decide the property which is topological property or not.

Definition 4.1. If a surmise mapping $(X_1,\tau_1)\xrightarrow{f}(X_2,\tau_2)$ is bijective, open and continuous, the surmise mapping f is called a homeomorphism and the topological knowledge spaces (X_1,τ_1) and (X_2,τ_2) are then said to be homeomorphic. On the other hand, a homeomorphism is a continuous bijective surmise mapping $(X_1,\tau_1)\xrightarrow{f}(X_2,\tau_2)$ whose inverse is continuous. If a surmise mapping f is a homeomorphism, it provides the following conditions,

the surmise mapping f is bijective.

i) a) $x_1 \neq y_1 \Rightarrow f(x_1) \neq f(y_1)$

b) $x \in X_2 \Rightarrow f^{-1}(x) \neq \phi$

the surmise mapping f is open.

ii) $A \in \tau_1 \Rightarrow f(A) \in \tau_2$

the surmise mapping f is continuous.

iii) $B \in \tau_2 \Rightarrow f^{-1}(B) \in \tau_1$.

Let the topological knowledge spaces (X_1, τ_1) and (X_2, τ_2) be homeomorphic. If the topological knowledge spaces (X_2, τ_2) and (X_1, τ_1) have got a property P at the same time, such property is called topological property.

Theorem 4.1. The well-gradedness is a topological property.

Proof. The theorem 3.1 states that τ_1 is a well-graded, if τ_2 is a well-graded and the surmise mapping $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ is continuous and bijective in the theorem 3.1. Henceforth, we can complete the proof if we show that the surmise mapping is open.

i) $A \in \tau_1 \Rightarrow f(A) \in \tau_2 \quad |A \Delta B| = n$

$f(A\Delta B) = f\,[(A \backslash B) \cup (B \backslash A)]$

$= f\,(A \backslash B) \cup f\,(A \backslash B)]$

$= f(A \cap B') \cup f(B \cap A')$

$= [f(A) \cap f(B')] \cup [f(B) \cap f(A')]$

$f(A) = K \in \tau_2, \qquad f(B) = L \in \tau_2$

$f(A') = K' \in \tau_2, \qquad f(B') = L' \in \tau_2$

$\Rightarrow [f(A) \cap f(B')] \cup [f(B) \cap f(A')] = (K \cap L') \cup (L \cap K')$

$=(K \backslash L) \cup (L \backslash K)$

$= |K \Delta L| = n$.

ii) $|A_{i-1} \Delta A_i| = 1, A_i \in \tau_1 \quad i = 1,2,\ldots,n$ and $\quad f(A_i) = L_i$,

$$f(A_{i-1} \Delta A_i) = f[(A_{i-1} \setminus A_i) \cup (A_i \setminus A_{i-1})] = \ldots$$

$= [f(A_{i-1}) \cap f(A_i')] \cup [f(A_i) \cap f(A_{i-1}')]$

$= (L_{i-1} \setminus \mathrm{L_i}) \cup (L_i \setminus \mathrm{L_{i-1}})$

$= |L_{i-1} \Delta L_i| = 1$.

It is obvious that when the surmise mapping f is an open surmise mapping, well-graded is preserved by it. Therefore, well-graded is a topological property.

Theorem 4.2. The learning smooth is a topological property.

Proof. The theorem 3.2 states that τ_1 is a learning smooth, if τ_2 is a learning smooth and the surmise mapping $(X_1,\tau_1)\xrightarrow{f}(X_2,\tau_2)$ is continuous and bijective in the theorem 3.2. In case of being an open mapping f, we need to show that learning smooth is preserved.

i) $A\in\tau_1 \Rightarrow f(A)\in\tau_2$.

Given $K,L\in\tau_1$ and for any two states $K\subseteq L$. There exists a tight path from K to L.

$K=A_0,A_1,...,A_n=L \quad |K\Delta L|=n$

$|A_{i-1}\Delta A_i|=1,\ \dot{I}=1,2,...,n$ and $A_0,A_1,...,A_n\in\tau_1$

$f(A_i)=B_i\in\tau_2$ (for f is an open mapping)

$$f(A_{i-1}\Delta A_i)=f[(A_{i-1}\setminus A_i)\cup(A_i\setminus A_{i-1})]$$
$$=f\left(A_{i-1}\cap A_i'\right)\cup f\left(A_i\cap A_{i-1}'\right)$$
$$=\left[f(A_{i-1})\cap f\left(A_i'\right)\right]\cup\left[f(A_i)\cap f\left(A_{i-1}'\right)\right]$$
$$=\left(B_{i-1}\cap B_i'\right)\cup\left(B_i\cap B_{i-1}'\right)$$
$$=(B_{i-1}\setminus B_i)\cup(B_i\setminus B_{i-1})$$
$$=B_{i-1}\Delta B_i$$

$|A_{i-1}\Delta A_i|=1 \quad\Rightarrow\quad |B_{i-1}\Delta B_i|=1$

$|B_0\Delta B_n|=n \quad K'=B_0,B_1,...,B_n=L'$

$K'=f^{-1}(K)\subseteq f^{-1}(L)=L'$.

For any two states $K'\subseteq L'$, there exists $K',L'\in\tau_2$. Therefore, it shows that there exists a tight path from K' to L'. As a result of this τ_2 is a learning smooth. Now that, an open surmise mapping f preserves the learning smoothness so learning smooth is a topological property.

Theorem 4.3. Accessible is a topological property.

Proof. The Theorem 3.3 states that τ_1 is accessible if τ_2 is accessible and the surmise mapping $(X_1,\tau_1)\xrightarrow{f}(X_2,\tau_2)$ is continuous and bijective in the theorem 3.3. For being complete the proof, in case of being an open mapping f, we need to show that accessible is preserved.

i) $A\in\tau_1 \Rightarrow f(A)\in\tau_2$,

given any $K\in\tau_1$ and $\phi\in\tau_1$. There exists a sequence of knowledge state like that

$\phi=K_0,K_1,...,K_n=K$.

$|K_n\Delta K_0|=n,\quad |K_{i-1}\Delta K_i|=1,\quad i=1,2,...,n$

$f(\phi)=f(K_0)=L_0=\phi\in\tau_2$ and $f(K_n)=f(K)=L_n=L\in\tau_2$

$f(K_{i-1}\Delta K_i)=L_{i-1}\Delta L_i=1$.

Thus, it is shown that for any $L\in\tau_2$ there exists a tight path from ϕ to L. Therefore τ_2 is accessible.

Accumulation Point

The concept of accumulation point shall help to define the other topological concepts on topological knowledge spaces. Some of them are dense of knowledge state, separable knowledge space. Using the concept of the accumulation point (or the limit point), we can determine an item which is an accumulation point. Thus, as students are correctly answering this item, this item is luminous.

Definition 5.1. In the topological space (X,τ) every open sets τ_i contains a is often called an open neighborhood or a neighborhood of a. It is defined by

$$N(a)=\{A|a\in A, \quad A\in\tau\}$$

Definition 5.2. In the topological space (X,τ), given $A\subset X$. Then a point x of X (which may or may not be a point of A) is called an accumulation point of A (or limit point of A) if every neighborhood of x contains at least one point $y\in A$ distinct from x (Kreyszig, 1989).

$$[\tau_i\in N(x)\Rightarrow(\tau_i\setminus\{\mathrm{x}\})\cap A\neq\phi]\Leftrightarrow \text{a point } x \text{ is an accumulation point of } A \quad (i=1,2,...,n)$$

Theorem 5.1. If every element of a knowledge state τ_i is an accumulation point, the topological knowledge space is not well-graded.

Proof. $K\cap(\tau_i\setminus\{\mathrm{x}\})\neq\phi$ for $\forall x\in K$ and $\forall\ \tau_i\in N(x)$.
We obtain that
$x,y\in\tau_i$, $x,y\in K$ and $\tau_i\in\tau$

$$\exists\ y\in\tau_i\wedge\quad y\in K\Rightarrow\begin{matrix}\exists\cup\tau_i\neq K\setminus\{\mathrm{x}\}\notin\tau,\exists\cap\tau_i\neq K\setminus\{\mathrm{x}\}\notin\tau\\ \exists\cup\tau_i\neq K\cup\{\mathrm{y}\}\in\tau, y\notin X\setminus \mathrm{K}\end{matrix}$$

It is explain that when the student's knowledge state is exactly items that she can answer correctly, she is not affected by topological knowledge space is well-graded.

DISCUSSION

In this study, we introduce the main theorems whose emphasis is the important of continuity in the modeling of students' learning. The fact that two topological knowledge spaces are equivalent to each other by the continuous mappings which allows well-graded. Thus, the continuity lets the two topological knowledge spaces provided a stronger pedagogical ground for their use in the application of knowledge structure theory. Theorem 2.10 in Doignon and Falmagne (1999) states that a knowledge space is well-graded if and only if all learning paths are gradations. Lemma 4.4 in Koppen (1998) states that a knowledge space is well-graded if and only if it is accessible. From the continuity we conclude that the theorem and the lemma invariably carry over to topological knowledge spaces each other. It can be searched that the continuity lets the accumulation point investigated on the topological knowledge spaces. It can be also searched that the existent of accumulation point is topological property on the homeomorphic spaces or not.

References

Cosyn, E., & Uzun, H. B. (2009). Note on two necessary and sufficient axioms for a well-graded knowledge space. *Journal of Mathematical Psychology*, *53*, 1, 40-42.

Doignon, J. P., & Falmagne, J. C. (1985). Spaces for the assessment of knowledge. *International Journal of Man-Machine Studies*, *23*, 175–196.

Doignon, J. P., & Falmagne, J. C. (1999). *Knowledge spaces*., Berlin, Heidelberg, NY: Springer-Verlag.

Falmagne, J. C., & Doignon, J. P. (1988). A Markovian procedure for assessing the state of a system. *Journal of Mathematical Psychology*, *32*, 232–258.

Falmagne, J. C. (1989). A latent trait theory via a stochastic learning theory for a knowledge space. *Psychometrika*, *53*, 283–303.

Falmagne, J. C. (1993). Stochastic learning paths in a knowledge structure. *Journal of Mathematical Psychology*, *37*, 489-512.

Koppen, M. (1998). On alternative representations for knowledge spaces. *Mathematical Social Sciences*, *36*, 127–143.

Kreyszig, E. (1989). *Introductory functional analysis with applications*. New York: Wiley Classics Library Edition Published.